Approximate evaluation of constants and functions with continued fractions

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Notation and conventions

Regular continued fractions will be given in the bracket notation, where the first number before the semicolon is the leading integer of the expansion:

$$[b_0; b_1, b_2, b_3, \ldots] = b_0 + \cfrac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \ldots}}}$$

Non-regular continued fractions will be written in one line starting with the leading integer and with large slashes between each partial numerator and denominator:

$$b_0 + a_1 \Big/ b_1 + a_2 \Big/ b_2 + a_3 \Big/ b_3 + \ldots = b_0 + \cfrac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}}$$

This is not a common notation, but large slashes are not in wide use for any other purpose. They allow to print continued fractions in one line and indicate visually that all that follows is in the denominator of a given partial fraction. The a_n are called partial numerators, the $b_n (n > 0)$ partial denominators. We will call b_0 the leading integer, which is not necessarily equal to the integer part of the value of the continued fraction.
An exclamation mark will denote a factorial as usual, and a double exclamation mark indicates the product of odd or even numbers only: $n!! = n \cdot (n-2) \cdot (n-4) \cdots (2 + (n \bmod 2))$. If a parametrisation in the following leads to a factorial of a negative or zero argument, its value is meant to be 1.

Approximate evaluation of continued fractions

Approximations to the value of a continued fraction are called approximants or convergents. They are fractions p_n/q_n obeying the following recursion formula for $n \geq 1$:

$$\begin{aligned} p_n &= b_n\, p_{n-1} + a_n\, p_{n-2} \\ q_n &= b_n\, q_{n-1} + a_n\, q_{n-2} \\ p_{-1} &= 1, \qquad q_{-1} = 0, \qquad p_0 = b_0, \qquad q_0 = 1 \end{aligned}$$

The recursion formula represents a kind of non-linear interpolation. Its result is always between the two previous approximants, provided a_n and b_n have the same sign. (For a proof, consider the derivative of the function $f(x) = ((1-x)\,a + x\,b)/((1-x)\,c + x\,d),\ x \in [0,1]$.) Therefore the approximants provide an interval subdivision scheme bracketing the limit value of the continued fraction. If two successive approximants agree about a certain number of digits, the approximation is accurate to that number of digits. The larger a b_n is compared to the corresponding a_n, the smaller the new bracketing interval is compared to its predecessor, and the faster the convergence.

Arithmetic with partial numerators and denominators

Don't reduce partial numerators and denominators against each other! This will result in a different continued fraction. However, you can simultaneously divide a_n, b_n and a_{n+1} by the same number (or multiply them by a number) without changing the continued fraction. This is how continued fractions containing rational numbers as partial numerators or denominators can be rewritten as continued fractions containing only integers.

Algebraic numbers

All quadratic numbers have periodic continued fraction representations, and all periodic continued fractions represent a quadratic number. (See the Silverman reference for more on this.) Some of them have especially simple continued fraction representations.

$$\sqrt{2} = [1; 2, 2, \ldots]$$

$$\tfrac{\sqrt{5}+1}{2} = [1; 1, 1, \ldots]$$

Transcendental numbers

The following equations list continued fraction expansions of transcendental mathematical constants and expressions containing them. It is restricted to expansions which have patterns, which allows their use in numerical approximations with arbitrary precision. All parameters are integers. The running variable n counts up by one for each repetition of the terms between ellipses $(\ldots)$.

$$\pi = 0 + 4\Big/1 + 1\Big/3 + 4\Big/5 + \ldots + n^2\Big/(2\,n+1) + \ldots$$

$$\left(\tfrac{(2k-1)!!}{k!}\right)^2 \tfrac{1}{2^{2k}}\,\pi = 0 + 4\Big/(4\,k+1) + 9\Big/(8\,k+2) + 25\Big/(8\,k+2) + \ldots$$
$$\ldots + (2\,n+1)^2\Big/(8\,k+2) + \ldots\,, \qquad k \in \mathbb{N}_0$$

$$\frac{4}{\pi} = 1 + 1\Big/2 + 9\Big/2 + 25\Big/2 + \ldots + (2\,n+1)^2\Big/2 + \ldots$$

$$\frac{\pi^2}{12} = 0 + 1\Big/1 + 1\Big/3 + 16\Big/5 + \ldots + n^4\Big/(2\,n+1) + \ldots$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2\,n, 1, \ldots]$$

$$\tfrac{1}{e-2} = 1 + 1\big/2 + 2\big/3 + 3\big/4 + \ldots + n\big/(n+1) + \ldots$$

$$e^{\frac{1}{k}} = [1; k-1, 1, 1, 3\,k-1, 1, 1, 5\,k-1, 1, 1, 7\,k-1, 1, \ldots]\,, \qquad k \in \{2, 3, \ldots\}$$

$$e^{\frac{1}{k}} = 0 + k\big/(k-1) + 1\big/2\,k + 1\big/1 + 1\big/2 + 1\big/(2\,k-1) + 1\big/1 + 1\big/4 + \\ + 1\big/(2\,k-1) + \ldots + 1\big/1 + 1\big/2\,n + 1\big/(2\,k-1) + \ldots \qquad k \in \{2, 3, \ldots\}$$

$$\frac{e^{\frac{2k}{\ell}} - 1}{e^{\frac{2k}{\ell}} + 1} = 0 + k\big/\ell + k^2\big/3\,\ell + k^2\big/5\,\ell + \ldots + k^2\big/(2\,n+1)\,\ell + \ldots\,, \qquad k, \ell \in \mathbb{Z}\backslash\{0\}$$

$$\log 2 = 0 + 1\big/1 + \frac{1}{2}\Big/1 + \frac{1}{6}\Big/1 + \frac{2}{6}\Big/1 + \frac{2}{10}\Big/1 + \ldots + \frac{k}{4\,k-2}\Big/1 + \frac{k}{4\,k+2}\Big/1 + \ldots$$

$$\tan 1 = 1 + 1\big/1 + 1\big/1 + 1\big/3 + 1\big/1 + 1\big/5 + \ldots + 1\big/1 + 1\big/(2\,n+1) + \ldots$$

Transcendental functions

Continued fractions can represent functions if the partial numerators and denominators are allowed to be functions too. For complex-valued functions as in the following, the function value at each point is thereby expressed as a continued fraction with complex partial numerators and denominators. The speed of convergence will depend on the function argument. This section presents only a few functions that are common or have simple representations; see the first two references for more.

$$\exp z = 1 + 2\,z\big/(2-z) + \frac{z^2}{6}\Big/1 + \frac{z^2}{60}\Big/1 + \ldots + \frac{z^2}{4\,(2\,n-3)\,(2\,n-1)}\Big/1 + \ldots$$

$$\log(1+x) = 0 + x\big/1 + x\big/2 + x\big/3 + 4\,x\big/4 + 4\,x\big/5 + \ldots \\ \ldots + n^2\,x\big/2\,n + n^2\,x\big/(2\,n+1) + \ldots\,, \qquad x \in (-1, \infty)$$

$$\log(1+z) = 0 + z\big/1 + \frac{z}{2}\Big/1 + \frac{z}{6}\Big/1 + \frac{2\,z}{6}\Big/1 + \frac{2\,z}{10}\Big/1 + \ldots \\ \ldots + \frac{n\,z}{2\,(2\,n-1)}\Big/1 + \frac{n\,z}{2\,(2\,n+1)}\Big/1 + \ldots\,, \qquad z \in \mathbb{C} \setminus (-\infty, -1]$$

$$\sqrt{\pi}\,\exp\left(z^2\right)\,\mathrm{erfc}(z) = 0 + 1\big/z + \tfrac{1}{2}\big/z + 1\big/z + \ldots + \tfrac{n}{2}\big/z + \ldots \qquad \mathrm{Re}\,z > 0$$

References

The information in this document was collected from the following sources:

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